

Sharp Fixed-Frequency Phase Complexity in Two-Dimensional Linear Systems with Two Independent Delays: A Rank-Controlled 2-4-8 Hierarchy

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Abstract

We study real two-dimensional linear systems with two independent discrete delays. At a prescribed frequency, the characteristic equation becomes a bidegree-two phase equation on the two-dimensional torus. We treat this equation as a fiber problem: regular first-phase fibers contain at most one unit-modulus second phase, whereas compatible singular fibers may contain two. We prove that every compatible singular first phase is at least a double zero of the cleared phase polynomial. This multiplicity compensates for the possible fiber doubling and yields a sharp bound for finite phase sets. Let r denote the number of full-rank delayed matrices among B and C . For each fixed positive frequency, the phase set is either infinite or contains at most 2^{r+1} points. Thus the maximal finite counts are 2, 4, and 8 when zero, one, or two delayed matrices are full rank. Explicit real matrix constructions attain all three bounds, proving sharpness. We distinguish this fixed-frequency count from the global number of kernel or stability-crossing curves. The results give an exact structural phase count for two-dimensional linear retarded systems with two independent delays.

Keywords: Delay Differential Equations; Two Independent Delays; Phase Fibers; Rank-Dependent Complexity; Stability Switching.

I. INTRODUCTION

Time delays may destabilize otherwise stable dynamical systems, generate repeated stability switches, and create highly structured critical sets in delay-parameter space. For systems with two independent delays, this structure has motivated several complementary approaches based on stability-crossing curves, D-subdivision, unit-circle parametrizations, CTCR, frequency sweeping, resultant elimination, and imaginary-spectrum bounds; see, among others, [1, 2, 3, 4, 5, 6, 7, 8]. Delay-independent stability under structural rank assumptions has also been studied, including the rank-one two-delay setting [9]. These works provide powerful tools for identifying critical frequencies and constructing stability boundaries.

Frequency-sweeping and pseudo-delay approaches have already counted solution pairs at prescribed frequencies as part of stability-chart construction [7, 8]; fixed-frequency counting by itself is therefore not claimed as new here. The question addressed here is structural: for the two-dimensional matrix-delay problem, how large can the fundamental phase set be at a prescribed frequency, and how do singular phase fibers affect that cardinality? Because the delays enter through $e^{-i\omega\tau_1}$ and $e^{-i\omega\tau_2}$, the natural object is the torus $\mathbb{T}^2$. The resulting fixed-frequency set is finite in generic situations but can become infinite under degeneracy. Our aim is to determine a sharp finite cardinality bound when the state dimension is two.

The two-dimensional determinant has a special algebraic structure. After the phase substitution, the characteristic condition becomes a bidegree-two equation.

$$F_\omega(z, w) = P + Qz + Rw + \beta z^2 + \gamma w^2 + \mu zw = 0, \quad (z, w) \in \mathbb{T}^2.$$

A first-phase elimination produces a Laurent equation of harmonic order at most four, suggesting a degree-eight bound after clearing negative powers. Degree alone, however, is not enough: at singular first phases one value of z may generate two distinct unit-modulus values of w . Counting first-phase zeros without treating this branching can therefore underestimate the number of phase pairs.

The central mechanism of this paper is a multiplicity-fiber compensation principle. We classify regular and singular phase fibers, prove that every compatible singular first phase is at least a double zero of the cleared phase polynomial, and show that this double multiplicity exactly compensates for the possible two-valued singular fiber. This converts algebraic degree into a bound on the actual number of phase pairs.

Define

$$r = 1_{\{\det B \neq 0\}} + 1_{\{\det C \neq 0\}}.$$

The quantity r is not the sum of the matrix ranks. It counts only how many of the two delayed matrices are full rank. Since $B, C \in M_2(\mathbb{R})$, full rank, invertibility, and nonzero determinant are equivalent. Our principal result is:

$$\mathcal{P}_\omega \text{ infinite or } \#\mathcal{P}_\omega \leq 2^{r+1}.$$

Thus the sharp maximal finite counts are 2, 4, and 8 for $r = 0, 1, 2$, respectively.

The contribution is not the introduction of unit-circle variables, polynomial elimination, or the observation that delayed-matrix rank affects algebraic complexity. Rank-dependent degrees already appear naturally in existing delay parametrizations and resultant-based methods [4]. Likewise, fundamental delay domains and periodic offspring constructions are classical in the CTCR/spectral-delay-space literature [5]. The contribution is instead a sharp structural cardinality theorem for the two-dimensional matrix-delay problem: the complete singular-fiber classification needed for the count, the singular multiplicity lemma, the multiplicity-fiber counting principle, and the sharp 2-4-8 hierarchy with explicit attaining constructions.

The paper is organized as follows. Section 2 derives the exact two-dimensional characteristic structure. Section 3 formulates the fixed-frequency torus problem. Section 4 classifies regular and singular fibers. Section 5 develops the Laurent reduction, multiplicity compensation, harmonic-order bounds, and degenerate identity case. Section 6 proves the rank-controlled bound.

Section 7 relates fundamental phases to delay offspring and crossing directions. Section 8 gives exact sharpness constructions and a full-rank stability-switching illustration. Section 9 summarizes a robust computational workflow, and Section 10 discusses scope and relation to existing methods.

➤ Characteristic Structure and Characteristic Equivalence

Consider

$$\dot{X}(t) = AX(t) + BX(t - \tau_1) + CX(t - \tau_2), X(t) \in \mathbb{R}^2,$$

With $A, B, C \in M_2(\mathbb{R})$ and $\tau_1, \tau_2 \geq 0$. The characteristic determinant is:

$$\Delta(\lambda; \tau_1, \tau_2) = \det(A - \lambda I + Be^{-\lambda\tau_1} + Ce^{-\lambda\tau_2}).$$

For the system (4),

$$\begin{aligned} \Delta(\lambda) = & \lambda^2 - \text{tr}(A)\lambda + \det(A) \\ & + [\text{tr}(\text{adj}(A)B) - \lambda \text{tr}(B)]e^{-\lambda\tau_1} \\ & + [\text{tr}(\text{adj}(A)C) - \lambda \text{tr}(C)]e^{-\lambda\tau_2} \\ & + \det(B)e^{-2\lambda\tau_1} + \det(C)e^{-2\lambda\tau_2} \\ & + \text{tr}(\text{adj}(B)C)e^{-\lambda(\tau_1+\tau_2)}. \end{aligned}$$

Proof. For any 2×2 matrices M, N , $\det(M + N) = \det M + \det N + \text{tr}(\text{adj}(M)N)$. Apply this identity successively to $A - \lambda I$, $Be^{-\lambda\tau_1}$, and $Ce^{-\lambda\tau_2}$, and use $\text{tr}(\text{adj}(A - \lambda I)B) = \text{tr}(\text{adj}(A)B) - \lambda \text{tr} B$.

Introduce

$$a = \text{tr} A, \quad a_0 = \det A, \quad b = \text{tr} B, \quad c = \text{tr} C,$$

$$b_0 = \text{tr}(\text{adj} A B), \quad c_0 = \text{tr}(\text{adj} A C), \quad \beta = \det B, \quad \gamma = \det C,$$

$$\mu = \text{tr}(\text{adj} B C).$$

Then

$$\begin{aligned} \Delta = & \lambda^2 - a\lambda + a_0 + (b_0 - b\lambda)e^{-\lambda\tau_1} + (c_0 - c\lambda)e^{-\lambda\tau_2} \\ & + \beta e^{-2\lambda\tau_1} + \gamma e^{-2\lambda\tau_2} + \mu e^{-\lambda(\tau_1+\tau_2)}. \end{aligned}$$

For 2×2 matrices,

$$\begin{aligned} \det(xA + yB + zC) = & x^2 \det A + y^2 \det B + z^2 \det C \\ & + xy[\text{tr} A \text{tr} B - \text{tr}(AB)] \\ & + xz[\text{tr} A \text{tr} C - \text{tr}(AC)] \\ & + yz[\text{tr} B \text{tr} C - \text{tr}(BC)]. \end{aligned}$$

Hence the characteristic quasipolynomial is determined by the nine scalar descriptors

$$\mathcal{I} = (\text{tr}A, \text{tr}B, \text{tr}C, \det A, \det B, \det C, \text{tr}(AB), \text{tr}(AC), \text{tr}(BC)).$$

Indeed,

$$b_0 = ab - \text{tr}(AB), \quad c_0 = ac - \text{tr}(AC), \quad \mu = bc - \text{tr}(BC).$$

If two triples (A, B, C) have identical nine descriptors (9), then their characteristic quasipolynomials (7) coincide for every $(\lambda, \tau_1, \tau_2)$. Consequently they have identical characteristic roots and stability maps.

This is a sufficient characteristic-equivalence statement; no claim is made that the nine descriptors classify triples under simultaneous similarity, nor that they form a minimal invariant set.

➤ Fixed-Frequency Phase Formulation

Fix $\omega > 0$ and set

$$z = e^{-i\omega\tau_1}, \quad w = e^{-i\omega\tau_2}, \quad |z| = |w| = 1.$$

Define

$$P = a_0 - \omega^2 - ia\omega, \quad Q = b_0 - ib\omega, \quad R = c_0 - ic\omega.$$

Substitution of $\lambda = i\omega$ into (7) yields

$$F_\omega(z, w) = P + Qz + Rw + \beta z^2 + \gamma w^2 + \mu zw = 0.$$

The fixed-frequency fundamental phase set is

$$\mathcal{P}_\omega = \{(z, w) \in \mathbb{T}^2 : F_\omega(z, w) = 0\}.$$

Whenever $\mathcal{P}_\omega$ is finite, including the case $\mathcal{P}_\omega = \emptyset$, define

$$\kappa_\omega = \#\mathcal{P}_\omega \in \mathbb{N}_0,$$

And call κ_ω the finite fundamental phase complexity at frequency ω .

Writing $z = e^{-i\theta_1}$ and $w = e^{-i\theta_2}$ with $(\theta_1, \theta_2) \in [0, 2\pi)^2$ gives a bijection between fundamental angle pairs and phase pairs. Thus κ_ω is also the number of fundamental angle pairs.

For fixed ω and $z \in \mathbb{T}$, define

$$\mathcal{W}_\omega(z) = \{w \in \mathbb{T} : F_\omega(z, w) = 0\}.$$

The counting problem reduces to understanding the possible cardinalities of these fibers and the number of first phases that support them.

➤ Regular and Singular Phase Fibers

For a fixed $z \in \mathbb{T}$, write

$$S(z) = P + Qz + \beta z^2, \quad D(z) = R + \mu z.$$

Equation (13) becomes

$$\gamma w^2 + Dw + S = 0.$$

Because $|w| = 1$, division by w gives

$$\gamma w + D + S\bar{w} = 0.$$

Taking conjugates and using real γ yields

$$\bar{S}w + \gamma\bar{w} = -\bar{D}.$$

Thus

$$\begin{pmatrix} \gamma & S \\ \bar{S} & \gamma \end{pmatrix} \begin{pmatrix} w \\ \bar{w} \end{pmatrix} = - \begin{pmatrix} D \\ \bar{D} \end{pmatrix}.$$

Define

$$\delta(z) = \gamma^2 - |S(z)|^2.$$

If $\delta(z) \neq 0$, the fiber $\mathcal{W}_\omega(z)$ contains at most one point. Its only possible element is

$$w = \frac{S\bar{D} - \gamma D}{\gamma^2 - |S|^2}.$$

Moreover this reconstructed value lies on $\mathbb{T}$ if and only if

$$\Phi_\omega(z) = |S\bar{D} - \gamma D|^2 - (\gamma^2 - |S|^2)^2 = 0.$$

Proof. Invert the matrix in (21); its determinant is $\delta(z)$. The first component is (23). Since the system (21) was obtained by imposing $\bar{w}$ as the conjugate of w , the remaining condition is $|w| = 1$, which is exactly (24). Therefore no extraneous unit-circle solution is introduced in the regular branch.

An equivalent form is

$$\Phi_\omega = (|S|^2 + \gamma^2)|D|^2 - 2\gamma\Re(\bar{S}D^2) - (\gamma^2 - |S|^2)^2.$$

➤ Singular fibers

Suppose first $\gamma \neq 0$ and $\delta(z) = 0$, so $|S| = |\gamma|$. The rank-one system (21) is compatible precisely when

$$\gamma\bar{D} = \bar{S}D.$$

Write

$$\frac{S}{\gamma} = e^{i\psi}.$$

Under compatibility,

$$\rho = -\frac{D}{2\gamma e^{i\psi/2}} \in \mathbb{R}.$$

The original quadratic has unit-circle roots iff $|\rho| \leq 1$, equivalently.

$$|D| \leq 2|\gamma|.$$

The roots are

$$w_{\pm} = e^{i\psi/2} \left(\rho \pm i\sqrt{1-\rho^2} \right).$$

Let $z \in \mathbb{T}$ satisfy $\delta(z) = 0$.

- If $\gamma \neq 0$ and compatibility (26) fails, then $\mathcal{W}_{\omega}(z) = \emptyset$.
- If $\gamma \neq 0$, compatibility holds, and $|D| < 2|\gamma|$, then $\#\mathcal{W}_{\omega}(z) = 2$.
- If $\gamma \neq 0$, compatibility holds, and $|D| = 2|\gamma|$, then $\#\mathcal{W}_{\omega}(z) = 1$.
- If $\gamma \neq 0$, compatibility holds, and $|D| > 2|\gamma|$, then $\mathcal{W}_{\omega}(z) = \emptyset$.
- If $\gamma = 0$, singularity implies $S = 0$. Then $Dw = 0$: if $D \neq 0$ there is no unit-circle solution, while if $D = 0$ every $w \in \mathbb{T}$ is admissible.

Thus a phase fiber may have cardinality 0, 1, 2, or ∞ .

Phase-fiber singularity $\gamma^2 - |S(z)|^2 = 0$, multiple characteristic roots $\Delta(i\omega) = \partial_{\lambda}\Delta(i\omega) = 0$, and geometric singularities of projected critical-delay curves are different notions and may occur independently.

➤ Laurent Phase Equation and Multiplicity Compensation

For a polynomial or Laurent polynomial G , define the involution

$$G^{\#}(z) = \overline{G(1/\bar{z})}.$$

On $\mathbb{T}$, $G^{\#}(z) = \overline{G(\bar{z})}$. Define

$$\begin{aligned} N(z) &= S(z)D^{\#}(z) - \gamma D(z), & \delta_L(z) \\ &= \gamma^2 - S(z)S^{\#}(z). \end{aligned}$$

Then on $\mathbb{T}$,

$$\Phi_{\omega}(z) = N(z)N^{\#}(z) - \delta_L(z)^2.$$

We call (33) the reduced Laurent phase equation. Multiplying by the minimal power z^m that clears negative exponents gives the cleared phase polynomial

$$H_{\omega}(z) = z^m \Phi_{\omega}(z).$$

Since unit-circle points satisfy $z \neq 0$, clearing negative powers does not alter multiplicities there.

Assume $\gamma \neq 0$ and let $z_{\star} \in \mathbb{T}$ be a compatible singular first phase.

Then

$$\text{ord}_{z_{\star}} \Phi_{\omega} \geq 2, \quad \text{mult}_{z_{\star}} H_{\omega} \geq 2.$$

Proof. Compatibility and singularity imply

$$N(z_{\star}) = N^{\#}(z_{\star}) = \delta_L(z_{\star}) = 0.$$

Because $z_{\star} \neq 0$, these Laurent polynomials are holomorphic in a neighborhood of $z_{\star}$. Hence there exist holomorphic functions n_1, n_2, d_1 such that

$$\begin{aligned} N(z) &= (z - z_{\star})n_1(z), & N^{\#}(z) \\ &= (z - z_{\star})n_2(z), & \delta_L(z) \\ &= (z - z_{\star})d_1(z). \end{aligned}$$

Therefore

$$\Phi_{\omega}(z) = (z - z_{\star})^2 [n_1(z)n_2(z) - d_1(z)^2],$$

Which proves the first inequality. Multiplication by z^m does not change the multiplicity at $z_{\star} \neq 0$. Any cancellation in the bracket can only increase the multiplicity.

Let $N_{\text{reg}}(\omega)$ denote the number of regular admissible first phases, and let $N_{\text{sing},1}(\omega)$ and $N_{\text{sing},2}(\omega)$ denote the numbers of compatible singular first phases with one and two admissible second phases, respectively.

Assume $\mathcal{P}_{\omega}$ is finite and $H_{\omega} \not\equiv 0$. Then

$$\#\mathcal{P}_{\omega} \leq \deg H_{\omega}.$$

Proof. The actual phase-pair count is

$$\#\mathcal{P}_{\omega} = N_{\text{reg}} + N_{\text{sing},1} + 2N_{\text{sing},2}.$$

Every regular admissible first phase contributes a zero of H_{ω} with multiplicity at least one. By Lemma 5.1, each compatible singular first phase contributes multiplicity at least two. Therefore

$$N_{\text{reg}} + 2N_{\text{sing},1} + 2N_{\text{sing},2} \leq \deg H_{\omega}.$$

Since $N_{\text{sing},1} \geq 0$, combining (37) and (38) proves (36).

➤ Harmonic Order

Let $d = \deg S \leq 2$. Since D has support $[0,1]$ and $D^{\#}$ has support $[-1,0]$,

$$\text{supp} N \subset [-1, d], \quad \text{supp} N^{\#} \subset [-d, 1],$$

So

$$\text{supp}(NN^{\#}) \subset [-d-1, d+1].$$

Also

$$\text{supp}(\delta_L^2) \subset [-2d, 2d].$$

With $d = \deg S \leq 2$,

$$\text{supp} \Phi_{\omega} \subset [-h, h], \quad h \leq \max\{d+1, 2d\}.$$

Consequently, if $\beta = 0$, then $\deg H_\omega \leq 4$, whereas if $\beta \neq 0$, then $\deg H_\omega \leq 8$. Moreover, if $\beta \neq 0$ and $P \neq 0$, the fourth harmonic is present and

$$\Phi_\omega = -\beta^2 \bar{P}^2 \neq 0.$$

Proof. Equations (39)–(40) prove (41). If $d = 2$, $NN^\#$ has harmonic order at most three, so the fourth harmonic of Φ_ω can arise only from $-\delta_L^2$. The coefficient of z^2 in $SS^\#$ is $\beta \bar{P}$, hence the coefficient of z^2 in δ_L is $-\beta \bar{P}$. Squaring and applying the minus sign in (33) gives (42).

➤ Identity-Degenerate Reductions

A degree bound cannot be used when $H_\omega \equiv 0$. The following proposition closes this loophole in the full-rank elimination branch.

Assume $\gamma \neq 0$. If

$$\Phi_\omega \equiv 0,$$

Then either $\mathcal{P}_\omega = \emptyset$ or $\#\mathcal{P}_\omega = \infty$. Consequently,

$$0 < \#\mathcal{P}_\omega < \infty \Rightarrow \Phi_\omega \not\equiv 0.$$

Proof. If $\delta_L \not\equiv 0$, it has only finitely many zeros on $\mathbb{T}$. At every other $z \in \mathbb{T}$, the regular reconstruction applies, and $\Phi_\omega(z) = 0$ produces a unit-modulus second phase. Hence infinitely many phase pairs exist.

Suppose therefore $\delta_L \equiv 0$. Then $|S(z)| = |\gamma|$ for every $z \in \mathbb{T}$. By Lemma B.1 in Appendix B, $S(z) = sz^m$ with $|s| = |\gamma|$ and $m \in \{0, 1, 2\}$. Since $\Phi_\omega \equiv 0$ and $\delta_L \equiv 0$, we have $NN^\# \equiv 0$. The Laurent-polynomial ring $\mathbb{C}[z, z^{-1}]$ is an integral domain, so $N \equiv 0$ or $N^\# \equiv 0$; the two are equivalent under the involution, hence $N \equiv 0$.

The three monomial cases are analyzed in Appendix B. In each case the compatible singular fibers are either inadmissible for every z or admissible for infinitely many z . Thus the phase set is empty or infinite.

➤ Sharp Rank-Controlled Fixed-Frequency Phase Complexity

We now prove the principal bound. The key is to select the elimination direction according to the rank class.

The case $\mathbf{r} = \mathbf{0}$

Here $\beta = \gamma = 0$, and

$$F_\omega(z, w) = P + Qz + Rw + \mu zw.$$

Define

$$U(z) = P + Qz, \quad V(z) = R + \mu z.$$

If $V(z) \neq 0$, then

$$w = -\frac{U(z)}{V(z)}.$$

And $|w| = 1$ is equivalent to

$$|P + Qz| = |R + \mu z|.$$

On $\mathbb{T}$ this becomes the first harmonic

$$K_0 + K_1 z + \bar{K}_1 \bar{z} = 0,$$

Where

$$K_0 = |P|^2 + |Q|^2 - |R|^2 - \mu^2, \quad K_1 = \bar{P}Q - \bar{R}\mu.$$

Multiplying by z gives

$$K_1 z^2 + K_0 z + \bar{K}_1 = 0.$$

If this polynomial is nonzero, it has at most two first phases. If it vanishes identically, then every regular first phase satisfies the unit-modulus condition, producing infinitely many phase pairs whenever any exist. If $V(z_*) = 0$, the original equation (45) reduces directly to $U(z_*) = 0$: when $U(z_*) \neq 0$ there is no pair above z_* , while when $U(z_*) = 0$ every $w \in \mathbb{T}$ is admissible and $\mathcal{P}_\omega$ is infinite. Thus a finite phase set has at most two points.

The cases $\mathbf{r} = \mathbf{1}$ and $\mathbf{r} = \mathbf{2}$

If $r = 1$, relabel the delays if necessary so that $\beta = 0$ and $\gamma \neq 0$. Then Lemma 5.3 gives $\deg H_\omega \leq 4$. If $\mathcal{P}_\omega$ is finite and nonempty, Proposition 5.4 excludes $H_\omega \equiv 0$. Lemma 5.2 therefore yields $\#\mathcal{P}_\omega \leq 4$.

If $r = 2$, both delayed matrices are invertible. Either elimination direction is available. With the present w -elimination, Lemma 5.3 gives $\deg H_\omega \leq 8$. Proposition 5.4 and Lemma 5.2 then yield $\#\mathcal{P}_\omega \leq 8$ whenever the phase set is finite.

Fix $\omega > 0$ and define r by (2). Then exactly one of the following alternatives holds:

$$\#\mathcal{P}_\omega = \infty,$$

Or

$$\#\mathcal{P}_\omega \leq 2^{r+1}.$$

Consequently, whenever $\mathcal{P}_\omega$ is finite,

$$\#\mathcal{P}_\omega \leq \begin{cases} 2, & r = 0, \\ 4, & r = 1, \\ 8, & r = 2. \end{cases}$$

Proof. The three rank classes were proved above. The empty set is included in the finite branch. Since an infinite set cannot satisfy a finite cardinality bound, the alternatives are mutually exclusive.

Table 1 Rank-Adapted Algebraic Hierarchy for the Fixed-Frequency Phase Problem.

$\mathbf{r}$	Full-rank delayed matrices	Maximal harmonic order	Maximal finite phase count
0	0	1	2
1	1	2	4
2	2	4	8

➤ *From Fundamental Phases to Critical-Delay Families*

Let a fundamental phase pair be written as

$$z = e^{-i\theta_1}, \quad w = e^{-i\theta_2}, \quad \theta_j \in [0, 2\pi).$$

The corresponding delay offspring are

$$\tau_1^{(k)} = \frac{\theta_1 + 2\pi k}{\omega}, \quad \tau_2^{(\ell)} = \frac{\theta_2 + 2\pi \ell}{\omega}, \quad k, \ell \in \mathbb{N}_0.$$

A fixed-frequency phase pair is therefore a fundamental critical generator. As ω varies along a local phase branch,

$$\begin{aligned} \tau_{1,j}^{(k)}(\omega) &= \frac{\theta_{1,j}(\omega) + 2k\pi}{\omega}, & \tau_{2,j}^{(\ell)}(\omega) \\ &= \frac{\theta_{2,j}(\omega) + 2\ell\pi}{\omega}. \end{aligned}$$

Theorem 6.1 bounds the number of generators at one frequency; it does not bound the total number of offspring points or the global number of stability-crossing curves.

For a simple imaginary characteristic root, $\partial_\lambda \Delta \neq 0$, the crossing sensitivity follows by implicit differentiation. From (7),

$$\begin{aligned} \partial_\lambda \Delta &= 2\lambda - a + [-b - \tau_1(b_0 - b\lambda)]e^{-\lambda\tau_1} \\ &\quad + [-c - \tau_2(c_0 - c\lambda)]e^{-\lambda\tau_2} - 2\tau_1\beta e^{-2\lambda\tau_1} \\ &\quad - 2\tau_2\gamma e^{-2\lambda\tau_2} - (\tau_1 + \tau_2)\mu e^{-\lambda(\tau_1 + \tau_2)}, \\ \partial_{\tau_1} \Delta &= -\lambda[(b_0 - b\lambda)e^{-\lambda\tau_1} + 2\beta e^{-2\lambda\tau_1} + \mu e^{-\lambda(\tau_1 + \tau_2)}], \\ \partial_{\tau_2} \Delta &= -\lambda[(c_0 - c\lambda)e^{-\lambda\tau_2} + 2\gamma e^{-2\lambda\tau_2} + \mu e^{-\lambda(\tau_1 + \tau_2)}]. \end{aligned}$$

Hence

$$\frac{\partial \lambda}{\partial \tau_j} = -\frac{\partial_{\tau_j} \Delta}{\partial_\lambda \Delta}.$$

Along a direction $v = (v_1, v_2)$ in delay space, the real crossing speed is

$$\sigma_v = \Re \left[-\frac{v_1 \partial_{\tau_1} \Delta + v_2 \partial_{\tau_2} \Delta}{\partial_\lambda \Delta} \right]_{\lambda=i\omega^*}.$$

We use the terms imaginary-root crossing and stability switching; no nonlinear Hopf conclusion is implied.

➤ *Exact Sharpness Constructions and Stability Switching*

All characteristic determinants in this section are obtained by direct expansion of $\det(A - iI + Bz + Cw)$, independently of the invariant representation used in the main derivation.

Sharpness for $\mathbf{r} = \mathbf{0}$

Take

$$A = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \omega = 1.$$

Both delayed matrices have determinant zero. Direct calculation gives

$$\begin{aligned} \det(A - iI + Bz + Cw) \\ = (-1 + 4i) + (-2 - i)z + 2w. \end{aligned}$$

The unit condition reduces to

$$18 - 4\cos\theta + 18\sin\theta = 0.$$

Under the half-angle substitution $t = \tan(\theta/2)$,

$$11t^2 + 18t + 7 = (t + 1)(11t + 7) = 0.$$

The two phase pairs are

$$\begin{aligned} (z_1, w_1) &= (i, -i), & z_2 &= \frac{36 + 77i}{85}, & w_2 \\ &= \frac{8}{17} - \frac{15}{17}i. \end{aligned}$$

Thus $\kappa_1 = 2$.

Sharpness for $\mathbf{r} = \mathbf{1}$

Take

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, & B &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & C \\ &= \begin{pmatrix} -3 & 31/10 \\ -100/31 & 3 \end{pmatrix}, & \omega &= 1. \end{aligned}$$

Then $\det B = 0$ and $\det C = 1$. Direct expansion gives

$$\begin{aligned} \det(A - iI + Bz + Cw) \\ = w^2 + \frac{100}{31}zw - \frac{1}{5}w - 2z - 2. \end{aligned}$$

With $z = (1 - it)/(1 + it)$, the reduced condition is

$$\Phi(t) = \frac{8(32242t^4 - 63283t^2 + 11675)}{24025(1 + t^2)^2}.$$

Setting $y = t^2$ gives

$$32242y^2 - 63283y + 11675 = 0,$$

Whose two positive roots are

$$y_{\pm} = \frac{63283 \pm 281\sqrt{31649}}{64484}.$$

Hence four real values of t occur. The singularity indicator reduces to

$$\delta(t) = \frac{t^2 - 15}{t^2 + 1},$$

And $t^2 = 15$ is not a root of (70). All four fibers are regular, so $\kappa_1 = 4$.

Sharpness for $\mathbf{r} = \mathbf{2}$

Take

$$A = \begin{pmatrix} 0 & \frac{1 + \sqrt{1441}}{40} \\ \frac{-1 + \sqrt{1441}}{40} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \omega = 1.$$

Now $\det B = -1$ and $\det C = 1$. Direct calculation gives

$$\det(A - iI + Bz + Cw) = w^2 + \frac{1}{20}w - z^2 - \frac{19}{10}.$$

Letting $x = \cos 2\theta$, the reduced condition becomes

$$\Phi(x) = -\frac{577600x^2 + 1096860x + 520343}{40000}.$$

Its roots are

$$x_{\pm} = \frac{-54843 \pm \sqrt{2253481}}{57760},$$

With

$$x_- \approx -0.975487532633, \quad x_+ \approx -0.923508312242.$$

Both lie strictly in $(-1, 1)$, and each generates four distinct first phases. The singular value $x = -19/20$ is not admissible because.

$$\Phi(-19/20) = \frac{39}{4000} \neq 0.$$

Thus $\kappa_1 = 8$.

For every $r \in \{0, 1, 2\}$ there exist real matrices $A, B, C \in M_2(\mathbb{R})$ and $\omega > 0$ such that the value of (2) equals r and

$$\#\mathcal{P}_{\omega} = 2^{r+1}.$$

Therefore

$$\max_{\substack{A, B, C \in M_2(\mathbb{R}), \omega > 0 \\ \mathbf{1}_{\{\det B \neq 0\}} + \mathbf{1}_{\{\det C \neq 0\}} = r \\ \#\mathcal{P}_{\omega} < \infty}} \#\mathcal{P}_{\omega} = 2^{r+1}, \quad r = 0, 1, 2.$$

Proof. The constructions (62), (67), and (73) attain 2, 4, and 8, respectively. Theorem 6.1 proves that these are upper bounds in the corresponding rank classes.

➤ A Full-Rank Stability-Switching Illustration

Consider

$$A = \begin{pmatrix} -0.96635405 & 0.34501060 \\ -0.46197656 & -0.62694617 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.65144293 & 0.01913992 \\ -1.56189071 & -0.36557530 \end{pmatrix}, \quad C = \begin{pmatrix} 0.08194363 & 1.49796946 \\ 0.44241191 & 0.58466164 \end{pmatrix}.$$

The displayed coefficients give

$$\det B = -0.2082569813294858, \quad \det C = -0.6148102328169154,$$

So both delayed matrices are full rank. At zero delay, the eigenvalues of $A + B + C$ are approximately

$$-0.32041366 \pm 1.71382986048i,$$

So the zero-delay system is stable. Along the ray

$$(\tau_1, \tau_2) = s(6.41821465, 6.21918700),$$

A refined imaginary-root crossing occurs at

$$s_c = 0.0267443911503931, \quad \omega_c = 1.5478762188157971,$$

With

$$\tau_1^* = 0.171651243086784, \quad \tau_2^* = 0.166328369765440.$$

Direct substitution of the refined critical values into the original characteristic determinant gives a residual at numerical precision. Nearby roots move from negative to positive real part along the ray, illustrating a genuine stability switch. This example is not used in the proof of Theorem 6.1.

➤ Numerical Implementation and Conditioning

The analytical theorem does not depend on numerical computation, but a robust implementation is

useful for reproducing examples and tracing critical-delay families. A practical workflow is:

- Precompute the scalar descriptors in (7);
- For each ω , compute P, Q, R ;
- Choose an elimination direction adapted to the delayed-matrix ranks;
- Construct the reduced Laurent phase equation and clear negative powers;
- Find candidate first phases on $\mathbb{T}$;
- Classify each candidate as regular or near-singular;
- Reconstruct regular second phases using (23);
- Classify singular fibers directly using Proposition 4.2;
- Verify every candidate in the original phase equation (13);
- Remove numerical duplicates and convert surviving phases to fundamental angles;
- Generate only the delay offspring needed for the intended stability region;
- Verify the original characteristic residual and, when needed, evaluate root simplicity and the crossing direction (61).

For w -elimination, a useful relative conditioning indicator is:

$$\rho_w = \frac{|\gamma^2 - |S|^2|}{\gamma^2 + |S|^2 + \varepsilon_{\text{mach}}}.$$

The dual elimination in Appendix D gives

$$\rho_z = \frac{|\beta^2 - |\tilde{S}|^2|}{\beta^2 + |\tilde{S}|^2 + \varepsilon_{\text{mach}}}.$$

These are diagnostics rather than mathematical definitions. Near a singular fiber, direct singular classification is preferable to dividing by a small reconstruction denominator.

The half-angle substitution

$$z = \frac{1 - it}{1 + it}$$

Is convenient for real-polynomial computation, but the omitted point $z = -1$ must always be tested separately. Numerical acceptance should verify $| |z| - 1|$, $| |w| - 1|$, $|F_\omega(z, w)|$, and, after conversion to delays, $|\Delta(i\omega; \tau_1, \tau_2)|$. No universal tolerance is prescribed because scaling varies with the matrix coefficients and frequency.

II. DISCUSSION

➤ Relation to Existing Two-Delay Methods

The literature on two independent delays has developed several complementary objects. Gu, Niculescu, and Chen [2] characterized stability-crossing curves for general two-delay systems. Naghnaeian and Gu [3] studied two scalar-delay channels with cross terms

and showed that elimination may create spurious crossing sets unless singular cases are handled carefully. Gao and Olgac [4, 5] developed imaginary-spectrum bounds and CTCR-based stability analysis for multiple delays, using half-angle transformations and resultant methods. Yuan et al. [6] derived an exact frequency-domain characterization of the imaginary spectrum for two-delay systems, while Hwang et al. [7] use resultant-based frequency sweeping to determine stability charts and count transformed solution pairs at prescribed frequencies, and Cai et al. [8] refine exhaustive stability-crossing-curve construction through a Jacobian criterion.

Accordingly, counting solution pairs at a prescribed frequency is not, by itself, the novelty claimed here. The phase-cardinality theorem considered here is complementary to these approaches. We do not seek the complete frequency range or the complete global crossing set. Instead, for one prescribed frequency, we ask how many torus points can satisfy the imaginary-root equation and how this finite count depends on the full-rank status of the delayed matrices.

➤ Why Polynomial Degree Alone does Not Solve the Counting Problem

The subtlety is the singular fiber. In a regular fiber, one admissible first phase produces at most one second phase, so the number of first-phase roots already controls the number of phase pairs. At a singular first phase, however, two unit-modulus second phases may coexist. A naive degree count can then fail. Lemma 5.1 shows that precisely such a compatible singular first phase has algebraic multiplicity at least two. Lemma 5.2 converts this observation into an exact accounting principle: the possible fiber doubling is paid for by double polynomial multiplicity.

➤ Meaning of the 2-4-8 Hierarchy

The hierarchy is controlled by

$$r = \mathbf{1}_{\{\det B \neq 0\}} + \mathbf{1}_{\{\det C \neq 0\}},$$

Not by the state dimension, which is fixed at two, and not by the sum of the ranks of B and C . The determinant terms βz^2 and γw^2 determine the maximal harmonic structure available to a rank-adapted elimination. The result is therefore a structural statement specific to two-dimensional systems with two independent delays. We do not extrapolate the formula 2^{r+1} to higher state dimensions or larger numbers of delays.

➤ Relation with Kernel and Offspring Representations

The use of a fundamental phase domain is closely related to kernel/offspring representations in the CTCR and spectral-delay-space literature [5]. The variables $z = e^{-i\omega\tau_1}$ and $w = e^{-i\omega\tau_2}$ encode the same periodicity at a prescribed frequency. The cardinality considered here is nevertheless different from the number of kernel curves: we count points of $\mathcal{P}_\omega \subset \mathbb{T}^2$ for one fixed ω , whereas a kernel curve is traced as frequency varies. Accordingly,

Theorem 6.1 is not a bound on the number of kernel curves or global stability-crossing curves.

➤ *Singular Fibers and Degeneracy*

Phase-fiber singularity should not be conflated with a multiple characteristic root. The former concerns failure of invertibility in the conjugate-linear reconstruction system for w ; the latter concerns $\partial_\lambda \Delta = 0$. The identity-degenerate case $\Phi_\omega \equiv 0$ is also qualitatively distinct: Proposition 5.4 shows that it cannot hide a nonempty finite phase set in the full-rank elimination branch. This closes a necessary logical gap in using polynomial degree as a counting device.

➤ *Characteristic Compression*

The nine descriptors (9) provide a convenient compression of the characteristic data in the two-dimensional setting. They can simplify symbolic manipulation and parameter studies because matrices with the same descriptors have identical characteristic quasipolynomials. This observation is secondary to the main theorem and is not presented as a complete simultaneous-similarity classification.

➤ *Scope and Limitations*

The result is a fixed-frequency theorem for real 2×2 retarded systems with two independent discrete delays. It does not count all critical-delay points in the unbounded delay plane, does not bound the global number of connected stability-crossing curves, and does not imply nonlinear Hopf bifurcation. The sharpness constructions establish optimality within this precise class. Extension to higher dimensions would require new structural information because the determinant produces more mixed delay harmonics and the singular-fiber geometry can be substantially more complicated.

➤ *Novelty Positioning*

Previous approaches already provide unit-circle, polynomial, resultant, CTCR, and frequency-domain techniques for locating imaginary roots and constructing stability boundaries; rank-dependent algebraic complexity also occurs in existing delay parametrizations. Within the two-delay literature examined in connection with these methods, including frequency-sweeping work that counts transformed solution pairs at prescribed frequencies, we have not found an explicit structural theorem giving the infinite- or 2^{r+1} dichotomy for the present 2×2 matrix-delay class, nor a proof through the singular-fiber multiplicity compensation developed here. The claim is intentionally limited to this precise combination of fixed-frequency cardinality, singular-fiber accounting, and sharp attainment.

III. CONCLUSION

For the two-dimensional two-delay system (4), the imaginary-root condition at a prescribed frequency reduces exactly to a bidegree-two phase equation on $\mathbb{T}^2$. Treating this equation as a fiber problem reveals a distinction between regular fibers, which contain at most

one admissible second phase, and singular fibers, which may contain zero, one, two, or infinitely many phases.

The principal technical result is that every compatible singular first phase is at least a double zero of the cleared phase polynomial. This multiplicity compensates for the possible two-valued singular fiber and converts the degree of a rank-adapted phase reduction into a genuine phase-pair count. If r is the number of full-rank delayed matrices, the fundamental phase set at every fixed $\omega > 0$ is therefore either infinite or has cardinality at most 2^{r+1} . Explicit real constructions attain 2, 4, and 8 for $r = 0, 1, 2$, proving that the hierarchy is sharp.

The theorem concerns fixed-frequency fundamental generators and is distinct from global crossing-curve counts. The phase-fiber viewpoint may nevertheless provide a useful local algebraic layer within broader stability-mapping and continuation procedures.

➤ *Complete Singular Phase-Fiber Classification*

For completeness, consider

$$\gamma w^2 + Dw + S = 0, \quad |w| = 1.$$

If $\gamma \neq 0$ and $|S| = |\gamma|$, write $S/\gamma = e^{i\psi}$. Compatibility of the singular conjugate-linear system is $\gamma \bar{D} = \bar{S}D$, which is equivalent to

$$De^{-i\psi/2}/\gamma \in \mathbb{R}.$$

Set $\rho = -D/(2\gamma e^{i\psi/2}) \in \mathbb{R}$. Substitution $w = e^{i\psi/2}u$ reduces (89) to

$$u^2 - 2\rho u + 1 = 0.$$

Thus

$$u_\pm = \rho \pm i\sqrt{1 - \rho^2}.$$

When $|\rho| < 1$, the two roots are distinct and lie on $\mathbb{T}$; when $|\rho| = 1$, they coalesce; and when $|\rho| > 1$, neither lies on $\mathbb{T}$. This proves the cases listed in Proposition 4.2. If $\gamma = 0$, singularity forces $S = 0$ and the equation is $Dw = 0$, giving the remaining two cases.

➤ *Identity-Degenerate Phase Equations*

Let $p(z) = \sum_{k=0}^n a_k z^k \neq 0$. If $|p(z)| = c > 0$ for every $z \in \mathbb{T}$, then

$$p(z) = az^m$$

For some m and $|a| = c$.

Proof. Let m and M be the smallest and largest indices with nonzero coefficients. Since $pp^\# = c^2$ is constant on $\mathbb{T}$, and hence as a Laurent polynomial, all nonzero harmonic coefficients must vanish. If $M > m$,

the coefficient of the highest Laurent power z^{M-m} is uniquely $a_M \bar{a}_m \neq 0$, a contradiction. Thus $M = m$.

Assume now $\gamma \neq 0$, $\delta_L \equiv 0$, and $\Phi_\omega \equiv 0$. By Lemma B.1, $S = sz^m$, $|s| = |\gamma|$, $m \in \{0,1,2\}$, and $N \equiv 0$.

If $m = 0$,

$$N = s\bar{R} + s\mu z^{-1} - \gamma R - \gamma\mu z,$$

So $\mu = 0$ and $s\bar{R} = \gamma R$. Every z is singular and compatible. If $|R| > 2|\gamma|$ the phase set is empty; otherwise infinitely many first phases are admissible.

If $m = 1$,

$$N = (s\bar{R} - \gamma\mu)z + (s\mu - \gamma R),$$

So $s\bar{R} = \gamma\mu$ and $s\mu = \gamma R$. With $\alpha = s/\gamma$, $|\alpha| = 1$,

$$R = \alpha\mu, \quad D(z) = \mu(\alpha + z).$$

If $\mu = 0$, $D \equiv 0$ and every first phase admits unit-circle roots. If $\mu \neq 0$, $D(-\alpha) = 0$, so by continuity an arc around $-\alpha$ satisfies $|D| \leq 2|\gamma|$, giving infinitely many admissible singular phases.

If $m = 2$,

$$N = s\bar{R}z^2 + \mu(s - \gamma)z - \gamma R,$$

So $R = 0$ and $\mu(s - \gamma) = 0$. If $\mu = 0$, $D \equiv 0$ and every first phase is admissible. If $\mu \neq 0$, then $s = \gamma$ and $D(z) = \mu z$, so $|D| = |\mu|$ for every z . The phase set is empty when $|\mu| > 2|\gamma|$ and infinite otherwise.

➤ Direct Symbolic Verification of the Sharpness Constructions

For the $r = 0$ construction, direct determinant expansion yields (63); substitution of both pairs in (66) gives zero exactly and all four displayed phases have unit modulus.

For $r = 1$, direct expansion gives (68). The regular reduction followed by $z = (1 - it)/(1 + it)$ yields (69). Its numerator is even, and the quadratic (70) has the two positive roots (71); hence four real values of t . The singular condition is (72), and $y = 15$ is not a root of (70). Therefore all four first phases reconstruct uniquely.

For $r = 2$, direct expansion yields (74). The reduced unit-circle equation depends on z^2 only through $x = \cos 2\theta$ and gives (75). Both roots (76) lie in $(-1,1)$, each producing four distinct values of θ modulo 2π . Equation (77) excludes the singular value. Thus exactly eight regular phase pairs occur.

➤ Dual Elimination and Characteristic Derivatives

Interchanging the roles of z and w , define

$$\tilde{S}(w) = P + Rw + \gamma w^2, \quad \tilde{D}(w) = Q + \mu w.$$

If

$$\tilde{\delta} = \beta^2 - |\tilde{S}|^2 \neq 0,$$

Then the regular reconstruction is

$$z = \frac{\tilde{S}\tilde{D} - \beta\tilde{D}}{\beta^2 - |\tilde{S}|^2}.$$

The corresponding unit-modulus reduction is obtained exactly as in Section 4. In the full-rank case, either elimination direction may be chosen; in computation, one may select the better-conditioned branch using (86) and (87).

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➤ Data Availability

No external dataset was used in this study. All theoretical results are analytical. The matrix coefficients and parameter values required to reproduce the numerical examples are reported in the manuscript.

➤ Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

➤ Author Contributions

All authors contributed equally to all stages of this work, including conceptualization, methodology, formal analysis, investigation, writing of the original draft, and review and editing of the manuscript.

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