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Homeomorphic Equivalences of Huye Museum Shapes as Topological Spaces: Analyzing Cultural Heritage Structures Through Topological Methods

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Abstract

In this work, we study the properties of Huye Museum Shapes that we regard as topological subspaces of the Euclidean plane $\mathbb{R}^2$. The properties under consideration are number of connected components and number of holes. We further look at the classification of these spaces by using the concepts of homeomorphism. Among the 50 considered spaces, we show that there are only 36 spaces that are not homeomorphic, this means that there are 36 equivalence classes with respect to homeomorphism equivalence. Each of those 50 spaces might be classified in one of those 36 distinct equivalence class.

I. INTRODUCTION

In Mathematics, mappings that preserve the structural aspect of a set play important roles. [6]. Mainly, they are used to map complicated sets into simpler or better-known ones to establish the properties of sets under consideration. In group theory, it is well known that isomorphisms are structure-preserving bijections between groups[12]. These kinds of mappings preserve all properties of groups, and two groups with an isomorphism between them are said to be isomorphic, and from a group theory point of view, there is no difference between such groups.

A similar concept in topology is the notion of homeomorphism, between topo- logical spaces. A homeomorphism is defined as a bijective continuous function between two topological spaces that has a continuous inverse function. Like iso- morphisms, homeomorphisms preserve all topological properties of spaces under which they are defined. Two spaces with a homeomorphism between them are said to be homeomorphic, and from a topological point of view, such spaces are similar[10].

Since being isomorphic is an equivalence relation, the concept of isomorphism in group theory is used to classify groups into equivalent ones. Similarly, h in topology are used to classify topological spaces.

Let X and Y be topological spaces. A bijective mapping $f : X \to Y$ is called a homeomorphism if f is continuous and the inverse function f^{-1} is continuous. The topological spaces X and Y are said to be homeomorphic if such a function exists. In this case, there is no topological difference between X and Y. Let us observe that being homeomorphic is reflexive, symmetric, and transitive and hence it is an equivalence relation. Accordingly, homeomorphisms can be used to classify a given collection of topological spaces into equivalence classes[1].

Homeomorphic spaces must have the same number of connected components and the same number of holes. However, spaces can have the same number of connected components and holes but not be homeomorphic[9]. In this project, we consider the collection of Huye museum shapes as topological spaces, The list of such spaces is presented at the end of this paper. The first problem considered here is to study the topological properties such as connectedness, path-connectedness, and to look at the number of invariant of those spaces. The second problem that is considered, is to classify those spaces by using the concepts of homeomorphism based on their topological invariants. We assume that these spaces have the subspace topology inherited from plane $\mathbb{R}^2$ with the Euclidean topology. However, the same classification can be made by considering $\mathbb{R}^2$ with another topology.

II. LITERATURE REVIEW

A homeomorphism is defined as a continuous bijective function with a continuous inverse, preserving the intrinsic "topological" properties of a shape such as connectedness, compactness, and the number of holes. Classic topology texts such as Munkres (2000) and Hocking Young (1961) have laid the groundwork by rigorously defining topological spaces, open sets, continuity, and homeomorphic transformations.

Two shapes are homeomorphic if they have the same number of connected components and holes. Edelsbrunner and Harer (2010) emphasized the use of these invariants in computational topology and shape recognition.

In computer vision, recognizing whether two shapes are homeomorphic is crucial for object classification, segmentation, and image registration. Zomorodian and Carlsson (2005) showed how topological fingerprints of shapes can be used to classify them robustly. Furthermore, Frosini and Landi (1999) developed size functions to compare shapes in a topologically meaningful way, applicable even when the shapes are not geometrically similar.

Recent research combines machine learning with topological invariants to learn shape similarity. The integration of deep learning with topological constraints (e.g., Hofer et al., 2019) has opened new avenues for shape classification and morphing. Moreover, homotopy-aware clustering and graph neural networks on simplicial complexes are active areas of exploration that extend the homeomorphism concept into data-driven shape analysis.

III. METHODOLOGY

In this study we use 50 photos taken from Huye museum fencing wall. There are so many shapes at Huye National museum, but some of them are continuous deformations of the others, and some of them look like they are equivalent. Our intention is to classify those image in their homeomorphism class based on their topological features.

➢ *Topological invariants*

- *Definition 3.1:*

A quantity I associated with topological spaces is a topological invariant if X and Y are homeomorphic implies that $I(X) = I(Y)$.
The number of vertices, (n-vertices, in fact, for $n \geq 3$), and the number of holes in the object are the topological invariants that we can identify.

➢ *Vertices*

The Number and type of vertices are the first topological invariant is in an object. Actually, a vertex is a point where multiple curves intersect or join together. The vertex type is determined by the number of intersecting curves[3].

- *Definition 3.2:*

An n-vertex in a subset L of a topological space X is an ele- ment $v \in L$ such that there exists some neighborhood $N_0 \subseteq X$ of v where all neighborhoods $N \subseteq N_0$ of v satisfy the following properties:
$N \cap L$ is connected.

The set formed by removing v from $N \cap L$, i.e., $\{a \in N \cap L | a \neq v\}$, is not connected, and is composed of exactly n disjoint sets, each of which is connected.

A set is connected if it is all in one piece. The above definition of n-vertex it means that if objects are close to vertex, it look like one component, and if we remove vertex from that component, then we get separate components each which is connected.

Since homeomorphism preserve connectedness, we say that the number of n- vertices is topological invariant for given $n \geq 3$. Thus, the connected set around a vertex must map to another connected set, and the set of n disjoint, connected pieces must map to another set of n-disjoint connected pieces[7].

In a nutshell any set close to a n-vertex is homeomorphic to any other set close to a n-vertex.

- *Example 3.3:*

Three curves intersecting in a 3-vertex are homeomorphic to any other three curves intersecting in a 3-vertex. However, they are not homeomorphic to a single curve.

➢ *Holes*

Topologically, a function mapping a space with a hole to one without a hole cannot be a homeomorphism. It is clear that a loop is not homeomorphic to a path with endpoints because it violates the continuity requirement of homeomorphism. So, number of hole is topological invariant[8].

➢ *Homeomorphism*

Homeomorphism is the most important concept in the field of topology because it preserve all properties given by a topology, and by that define a correspondence between points and between open sets in two topological spaces. Homeomorphism is defined as follow:

Let X and Y be two topological spaces, and let $f : X \rightarrow Y$ be a bijection with inverse $f^{-1} : Y \rightarrow X$. If both f and f^{-1} are continuous function, then f is said to be homeomorphism. If there exists homeomorphism between X and Y, we say that X and Y are homeomorphic or topologically equivalent, and we denote this by $X \cong Y$[4].

Let $f : X \to Y$ be a bijective function. For $f^{-1} : Y \to X$ to be continuous, it must be true that $(f^{-1})^{-1}(U)$ is open in Y for every open set U in X. But since f is bijection, $(f^{-1})^{-1}(U) = f(U)$ for $U \subset X$. Thus $f(U)$ must be open in Y for every open set in U in X. Therefore saying that f^{-1} is continuous when f is bijection is equivalent to saying that the image of every open set under f is an open set. Similarly, saying that f is continuous when f is bijection is equivalent to saying that the image of every open set under the inverse function, f^{-1}, is an open set[5].

We can say that f is homeomorphism if it is bijection on a points, and bi- jection on a collection of open sets making up the topology involved. Every point in X is matched to unique point in Y, with no point in Y is left over. At the same time, every open set in X is matched to unique open set in Y, with no open set in Y is left over.

- *Example 3.4:*

As spaces having the topology inherited from $\mathbb{R}^2$, the following topological spaces are homeomorphic:

Fig 1 Example of Homeomorphic Topological Spaces.

- *Example 3.5:*

As spaces having the topology inherited from $\mathbb{R}^2$, the following topological spaces are not homeomorphic. Indeed, the first space is made by one component while the second is made by five components. However, one can observe that each component in the second space is homeomorphic to the first space.

Fig 2 Example of Topological Spaces Which are Not Homeomorphic.

- *Theorem 3.6: Being Homeomorphic is an Equivalence Relation.*

The theorem 3.6 implies that homeomorphism satisfy the following properties:

- ✓ Refrexivity : X is homeomorphic to X.
- ✓ Symmetry : If X is homeomorphic to Y, then Y is homeomorphic to X.
- ✓ Transitivity : If X is homeomorphic to Y, and Y is homeomorphic to Z, then X is homeomorphic to Z

The resulting equivalence classes are called homeomorphism classes

IV. RESULT

- *Topological invariant of Huye museum shapes as topological subspaces of the plane*

The topological invariant that are considered here are the number of holes and the number of connected components.

- *Number of holes*

The number of holes of topological spaces is one among the topological invariant that can be used in the classification of homeomorphic topological spaces. For instance a circle, a square, and a rectangle, taken as subspaces of $\mathbb{R}^2$ with the Euclidean topology, are topologically equivalent. However, no one of them is topologically equivalent to a disc. According to the number of holes that is observed in each museum shape, taking them as topological spaces, they are classified as follows:

- ✓ 1 Hole: X_{30}.
- ✓ 3 Holes: X_{35}.
- ✓ 4 Holes: X_{14}, X_{18}, X_{27} and X_{34}.
- ✓ 5 Holes: X_{13}, X_{17}.
- ✓ 6 Holes : X_7.
- ✓ 8 Holes: X_{41}.
- ✓ 9 Holes: X_{26}.

All other spaces which are not appearing on this list have no holes, and no topological space having more than 9 holes.

- *Number of connected components*

As discussed in the previous sections, components of a topological spaces are the maximal connected subsets in such a space. According, if a topological space is connected, it then has one component. This implies that a topological space with more than one component must be a disconnected topological space[2].

By looking at Huye museum shapes as topological subspaces of $\mathbb{R}^2$, and an- alyzing them, according to the number of connected components, these spaces can be classified as follows:

- ✓ 2 Connected components: X_{12}, X_{21}, X_{29}, X_{33}, X_{40}, X_{41}, X_{48} and X_{50}.
- ✓ 3 Connected components: X_9, X_{15}, X_{23} and X_{44}.
- ✓ 4 Connected components: X_1, X_3, X_4, X_{11}, X_{27}, X_{46} and X_{49}.
- ✓ 5 Connected components: X_{25}.
- ✓ 6 Connected components: X_{17}.
- ✓ 7 Connected components: X_5.
- ✓ 9 Connected components: X_{20} and X_{45}.
- ✓ 12 Connected components: X_2 and X_{22}.
- ✓ 16 connected components: X_{37}.
- ✓ 17 Connected components: X_{43}.
- ✓ 18 Connected components: X_{47}.

All the spaces which are not mentioned on this list have only one component and hence they are connected. No topological space having more than 18 connected components.

- *Homeomorphic Spaces*

One of the most important concept is topological spaces, is the notion of homeo- morphisms. A homeomorphism or topological isomorphism is the most important concept in the field of topology because it preserve all properties given by a topology. It is defined as a bijective continuous function between topological spaces, and having a continuous inverse. If two topological spaces X and Y are homeomorphic; it means that there is a homeomorphism between them, then topologically, there is not difference between them.

Since homeomorphisms preserve all topological properties between spaces, the concepts of connectedness, number of holes, connected components, cutting sets, and so on, can be used to study homeomorphism in a class of spaces. For instance, if X is connected and Y is disconnected, the two spaces can not be homeomorphic. If X has one holes and Y has more than one holes, the two spaces can not be topologically equivalent[11].

In the following we identify topological equivalent spaces, by using connect- edness, number of holes, the number of connected components and other other techniques such as the cutting sets (points), in Huye museum shapes taken as topological subspaces of the plane. We point out that the concept of distance between points (or sets), the property of being bounded, the size (length, area, and volume), and the angles between objects, are meaningless in topological point of view. Hence, these concepts are not preserved by homeomorphism, since they are not defined by using open sets. Taking into consideration, the theory about homeomorphism, we can classify Huye museum shapes as topological spaces into homeomorphic equivalences.

Since the number of connected components is a topological property, and hence preserved by homeomorphisms, meaning two homeomorphic spaces must have the same number of connected components, it follows that [X_{17}], [X_5], [X_{25}], [X_{37}], [X_{43}] and [X_{47}] are equivalence classes. Each of them contains no other elements.

Since the number of holes that a topological space contains it is a topological property, and hence preserved by homeomorphisms, it follows that [X_{30}], [X_{35}], [X_7],[X_{26}] and [X_{41}] are equivalence classes. Each of them contains no other elements.

The spaces X_{13} and X_{17} have the same number of holes, but they have a different number of connected components. Hence they can not be homeomorphic, and thus [X_{13}], and [X_{17}] are different equivalence classes, each containing no other elements.

The spaces X_2 and X_{22} have the same number of connected components, and each component from X_2 is homeomorphic to each component from X_{22}. Hence the two spaces are topologically equivalent, and thus [X_2, X_{22}] is an equivalence class.

The spaces X_{20} and X_{45} have the same number of connected components. One component of X_{45} is a point, which is not the case for X_{20}, and thus they are not homeomorphic. It follows that each of them forms its own class. Hence [X_{20}] and [X_{45}] are equivalence classes, each containing no other elements.

The spaces X_{14}, X_{18}, X_{27} and X_{34} have the same number of holes, but no one is homeomorphic to another. Indeed, removing the middle solid square in X_{14}, we remain with a topological space connected with one hole. However, there is no solid square which can be removed from X_{18} and X_{34} to remains with a connected topological space with one hole. Such a square will leave X_{14} with at least two holes. Hence X_{14} is homeomorphic to any of X_{18} and X_{34}. Each horizontal line leaves two connected components for X_{34}, but there are horizontal line which give more that two connected components. Hence X_{18} and X_{34} are not homeomorphic. The space X_{27} is not homeomorphic to any of X_{14}, X_{18} and X_{34} since it has four connected components which is not the case for the others. Accordingly, we have [X_{14}], [X_{18}], [X_{27}] and [X_{34}] as equivalence classes, containing no other elements.

The spaces X_9, X_{15}, X_{23} and X_{44} have the same number of connected components. As one can observe, any two components from different spaces are homeomorphic and thus the spaces belong to the same equivalence class, that we denote by [X_9, X_{15}, X_{23}, X_{44}].

The spaces X_1, X_3, X_4, X_{11}, X_{27}, X_{46} and X_{49} have the same number of connected components. In above we have observed that X_{27} forms its own class, and thus it not homeomorphic to any of those spaces. We observe that the spaces X_1, X_4, X_{46} and X_{49} are homeomorphic. They are made by homeomorphic components, and thus they belong to the same equivalence class. No one of them is homeomorphic to X_{11}, since each component for X_{11} is like a circle, which is not the case for the others. We observe also that the space X_3 is not homeomorphic to any of these spaces. Hence we have the equivalence classes [X_1, X_4, X_{46}, X_{49}], [X_{11}], and [X_3].

The spaces $X_{12}, X_{21}, X_{29}, X_{33}, X_{40}, X_{41}, X_{48}$ and X_{50} have the same number of con- nected components. We observe directly that that the spaces X_{12}, X_{21}, X_{33}, X_{40} and X_{48} are homeomorphic, since any two components from different spaces are homeomorphic. No one of them is equivalent to X_{29} or X_{41}. The spaces X_{41} is not homeomorphic to X_{29}, since X_{29} has no holes while X_{41} have holes. Accordingly, we have the classes [X_{12}, X_{21}, X_{33}, X_{40}, X_{48}], [X_{29}] and [X_{41}].

The remaining non-classified spaces are X_6, X_8, X_{10}, X_{16}, X_{19}, X_{24}, X_{28}, X_{31}, X_{32}, X_{36}, X_{38}, X_{39} and X_{42}, and they are connected without holes. We observe that the spaces X_6, X_{10}, X_{24}, and X_{39} are like a section of a continuous curve in the plane. Hence they are homeomorphic, and accordingly, [X_6, X_{10}, X_{24}, X_{39}] is an equivalence class.

To classify the spaces X_8, X_{16}, X_{19}, X_{28}, X_{31}, X_{32}, X_{36}, X_{38} and X_{42}, we study two by two.

The spaces X_{38} and X_{42} are not homeomorphic. Indeed, each horizontal line leaves two connected components for X_{39} but there are horizontal lines which give more than three connected components for X_{42}.

The spaces X_{36} and X_{38} are not homeomorphic. Indeed, five points removed in X_{38} can leave at most four connected components, while in X_{36}, there are five points which can be removed and remain with six connected components.

The spaces X_{32} and X_{36} are not homeomorphic. Indeed, no vertical line leaves more than four connected components for X_{36}, but there are vertical lines which leave five connected components for X_{32}. A similar reasoning (by looking cutting sets), we observe that no spaces in the pairs X_{32} and X_{31}, X_{31} and X_{28}, X_{28} and X_{19}, X_{19} and X_{19}, X_{16} and X_8 are homeomorphic. Hence each of them forms its own class, and thus we the equivalences classes [X_8], [X_{16}], [X_{19}, [X_{28}], [X_{31}], [X_{32}], [X_{36}], [X_{38}] and [X_{42}] containing no more elements.

In summary, homeomorphic equivalence classes in Huye museum shapes regarded as topological subspaces of the plane, are the following:

31 Equivalences classes with one element: [X_{17}], [X_5], [X_{25}], [X_{37}], [X_{43}], [X_{47}], [X_{30}], [X_{35}], [X_7],[X_{26}], [X_{41}], [X_{13}], [X_{20}], [X_{45}], [X_{14}], [X_{18}], [X_{27}],

[X_{34}], [X_{11}], [X_3], [X_{29}], [X_{41}], [X_8], [X_{16}], [X_{19}, [X_{28}], [X_{31}], [X_{32}], [X_{36}], [X_{38}] and [X_{42}].

1 Equivalence classes with two elements: [X_2, X_{22}].

3 Equivalence classes with four elements: [X_9, X_{15}, X_{23}, X_{44}], [X_1, X_4, X_{46}, X_{49}] and [X_6, X_{10}, X_{24}, X_{39}].

1 Equivalence class with five elements: [X_{12}, X_{21}, X_{33}, X_{40}, X_{48}].

V. DISCUSSION

We have 36 equivalence classes, and accordingly, among the 50 topological spaces considered, only 36 spaces are not equivalent in the topological point of view. If two spaces are in the same equivalence class, then no difference between them in topological point of view. This means that if a space is making its own class, without any other one, then such a space is topologically different to the others.

Since homeomorphisms preserve all topological properties, it follows that all topological spaces that are classified in the same class have the same topological properties. Accordingly, to study the properties of spaces in one class, it enough to study one space, and take the conclusion on all other spaces belonging in that class.

VI. CONCLUSION

In this work, we have studied the topological properties of Huye museum shapes, that we take as topological spaces in the plane. The main properties which took into consideration were connectedness, compactness and path connectedness. We have also calculated different topological invariant for those spaces, such as the number of holes and the number of connected components, observed for each space.

The main problem which was under consideration was the classification of those spaces (regarded as topological subspaces of the Euclidean plane $\mathbb{R}^2$) into equivalent classes by using the concept of topological invariants. Among the 50 topological spaces considered in this work, we found that some of them are similar (from the topological point of view). We found that there are 36 equivalence classes in terms of homeomorphism, which means that there are only 36 non- homeomorphic spaces. The concept of homeomorphism is extremely important in the field of topology because is the correct and exact way of knowing the equality of topological spaces. If two spaces are homeomorphic, they have exactly the same topological properties and they are indistinguishable.

RECOMMENDATIONS

This section is dedicated to the recommendations to be taken into consideration for improving future research-oriented in the same area.

We have classified Huye museum shapes by using the concepts of homeo- morphism. However, there are many concepts in the topology that can be used to classify topological spaces. I recommend that in further study about this topic, different topological concepts beyond homeomorphism can be taken into consideration.

Nowadays, advanced machine learning algorithms are being used to capture hidden patterns in images. I recommend using a machine learning algorithm capable of capturing similarities in the topological invariants extracted in this study.

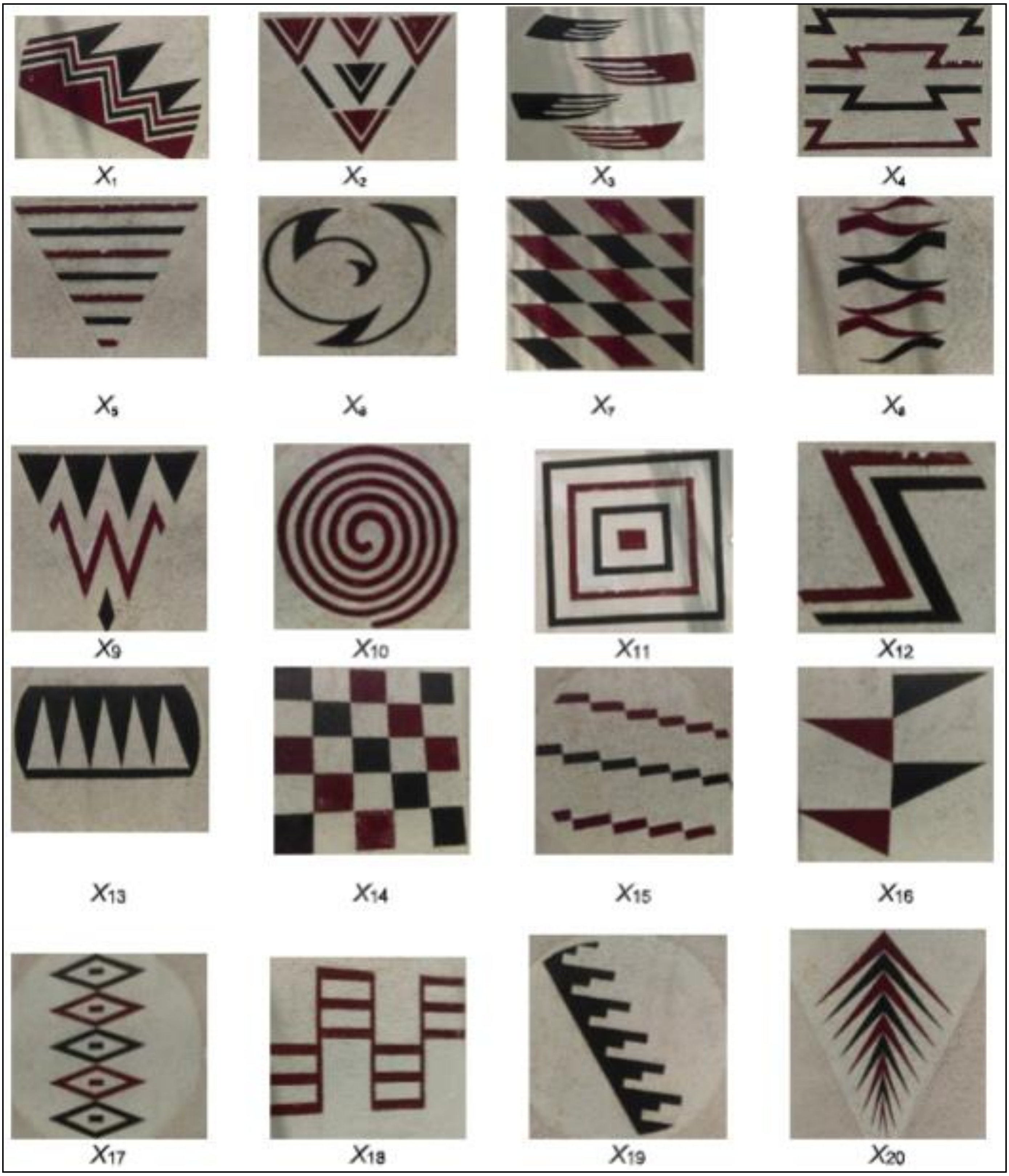
X_1 X_2 X_3 X_4
X_5 X_6 X_7 X_8
X_9 X_{10} X_{11} X_{12}
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Fig 3 Huye Ethnographic Shapes

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